INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2003; **43**:1099–1106 (DOI: 10.1002/fld.546)

A spatially adaptive linear space-time finite element solution procedure for incompressible flows with moving domains

Y. T. Feng*,† and D. Perić‡

Civil & Computational Engineering Centre, School of Engineering, University of Wales Swansea, Swansea, SA2 8PP, U.K.

SUMMARY

A linear solution strategy for the finite element simulation of incompressible fluid flows with moving domains is outlined in the context of a fully Lagrangian space-time GLS formulation using low-order elements. This linear solution strategy is achieved by assuming that the incompressibility condition is enforced although it is relaxed in the GLS formulation. The approach has a distinct advantage over the non-linear Newton–Raphson solution approach in a sense that it can not only significantly reduce the computing costs in terms of computer CPU time and memory requirements but also preserve the solution accuracy if a sufficiently small time-step size is applied. Its applicability is further demonstrated through a wave propagation and breaking problem. For this type of problems, adaptive re-meshing techniques are essential to achieve a successful simulation. A mesh adaptive procedure developed earlier for simulation of large deformation solid mechanics problems is appropriately modified and employed in simulation of flows of incompressible fluids with moving domains. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: Galerkin/least-squares scheme; Lagrangian formulation; incompressible Navier–Stokes flows; moving domain; solution strategy; adaptive remeshing

1. INTRODUCTION

Considerable attention has been paid to the numerical simulation of fluid flow problems involving changing spatial domains due to their great practical importance. In the last decade or so, the *space-time Galerkin/least-squares* (GLS) finite element method has been established as a general numerical approach to solve a wide variety of incompressible fluid flow problems including moving domains and free surfaces [1–6]. The main feature of the method is that the corresponding variational formulation employs the time-discontinuous Galerkin method and includes least-squares terms, which involve residual of the Euler–Lagrangian equations evaluated over element interiors, to stabilize the formulation. This stabilization nature of the

* Correspondence to: Y. T. Feng, Civil & Computational Engineering Centre, School of Engineering, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, U.K.
† E-mail: y.feng@swansea.ac.uk
‡ E-mail: d.peric@swansea.ac.uk

Contract/grant sponsor: EPSRC, U.K.; contract/grant number: GR/L66724

Received 2 April 2002
Revised 19 February 2003

formulation not only prevents numerical oscillation for incompressible flows using equal-order interpolation functions for velocity and pressure without enforcing the incompressibility constraint condition, but also preserves the consistency of the standard Galerkin method. It should be observed that the total volume change occurred in the numerical simulation depends on the choice of the values of the stabilization parameters involved and hence their values must be specified with care.

In our previous work [7], two solution strategies and a time adaptive scheme are proposed for the solution of incompressible Lagrangian fluid flow problems by employing the space-time Galerkin/least-squares method using low-order finite elements in both spatial and time domains. In particular, a linear solution procedure is suggested and numerically verified by several examples. This linear approach has a distinct advantage over the non-linear Newton–Raphson solution approach in a sense that it can not only significantly reduce the computing costs in terms of computer CPU time and memory requirements but also preserve the solution accuracy if a sufficiently small time-step size is applied.

In this paper, this linear solution strategy is highlighted and its applicability to a problem with moving domain is further demonstrated. In addition, an adaptive re-meshing technique, which is essential to achieve a successful simulation of the problem considered, is described.

2. SPACE-TIME GLS FINITE ELEMENT FORMULATION

2.1. Governing equations

Consider an incompressible viscous flow occupied a time-dependent spatial domain $\Omega(t) \in \mathbb{R}^{n_{\dim}}$, where $n_{\dim}$ is the number of space dimensions, with boundary $\Gamma(t)$. The flow is governed by the following set of equations with velocity $\mathbf{u}(\mathbf{x},t)$ and pressure $p(\mathbf{x},t)$ as primary variables:

$$\begin{array}{lll}
\text{Momentum equation:} & \rho\left(\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) - \nabla\cdot\boldsymbol{\sigma} = \mathbf{f} & \text{on } \Omega(t),\ \ \forall t \in [0,T] \\
\text{Incompressibility condition:} & \nabla\cdot\mathbf{u} = 0 & \text{on } \Omega(t),\ \ \forall t \in [0,T] \\
\text{Boundary conditions:} & \mathbf{u} = \mathbf{g} & \text{on } \Gamma_g(t), \\
 & \boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{h} & \text{on } \Gamma_h(t), \\
\text{Initial condition:} & \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0 & \text{on } \Omega(0),
\end{array} \tag{1}$$

where ρ is the density of the fluid, $\mathbf{g}$ and $\mathbf{h}$ are given functions and $\mathbf{n}$ is the unit outward normal vector of the boundary. The strain rate and stress tensors, $\boldsymbol{\varepsilon}(\mathbf{u})$ and $\boldsymbol{\sigma}$, are respectively defined as

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^t\right) \tag{2}$$

$$\boldsymbol{\sigma}(p,\mathbf{u}) = -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) \tag{3}$$

where μ is the viscosity of the fluid and $\mathbf{I}$ is the identity tensor.

As the problem with a changing domain is considered, the motion of the boundary is unknown in advance and thus the geometry of the domain $\Omega(t)$ is a part of the problem solution to be considered.

2.2. Variational formulation

Assume that the initial domain of the problem concerned is spatially discretized into elements Ω_0^e and the time interval $[0, T]$ is also partitioned into subintervals $I_n = [t_n, t_{n+1}]$. With $\Omega_n = \Omega(t_n)$ and $\Gamma_n = \Gamma(t_n)$, the space-time stab Q_n is defined as the domain enclosed by the surfaces Ω_n, Ω_{n+1} and B_n, where B_n is the surface described by the boundary $\Gamma(t)$ as t traverses I_n.

Suppose that $\mathscr{S}_n, (\mathscr{S}_p)_n, \mathscr{V}_n$ and $(\mathscr{V}_p)_n$ are all properly defined function spaces (see e.g. Reference [3] for their definitions) and let $\mathbf{u}_n^{\pm}$ be defined as

$$\mathbf{u}_n^{\pm} = \lim_{\varepsilon \to 0} \mathbf{u}(t_n \pm \varepsilon)$$

The variational formulation for the space-time Galerkin/least-squares (omitting the boundary related terms) can then be expressed as follows: given $\mathbf{u}_n^-$, find $\mathbf{u}_n \in \mathscr{S}_n$ and $p_n \in (\mathscr{S}_p)_n$ such that $\forall \mathbf{w} \in \mathscr{V}_n$ and $\forall q \in (\mathscr{V}_p)_n$

$$\begin{aligned}
&\int_{Q_n} \mathbf{w} \cdot \left[\rho\left(\frac{\partial \mathbf{u}_n}{\partial t} + \mathbf{u}_n \cdot \nabla \mathbf{u}_n\right) - \mathbf{f}\right] \mathrm{d}Q + \int_{Q_n} \boldsymbol{\varepsilon}(\mathbf{w}) : \boldsymbol{\sigma}(p_n, \mathbf{u}_n)\, \mathrm{d}Q + \int_{Q_n} q \nabla \cdot \mathbf{u}_n \, \mathrm{d}Q \\
&+ \sum_{e=1}^{n_{\mathrm{el}}} \int_{Q_n^e} \delta_1 \left[\rho\left(\frac{\partial \mathbf{w}}{\partial t} + \mathbf{u}_n \cdot \nabla \mathbf{w}\right) - \boldsymbol{\sigma}(q, \mathbf{w})\right] \cdot \left[\rho\left(\frac{\partial \mathbf{u}_n}{\partial t} + \mathbf{u}_n \cdot \nabla \mathbf{u}_n\right) - \boldsymbol{\sigma}(p_n, \mathbf{u}_n) - \mathbf{f}\right] \mathrm{d}Q \\
&+ \sum_{e=1}^{n_{\mathrm{el}}} \int_{Q_n^e} \rho \delta_2 \nabla \cdot \mathbf{w} \nabla \cdot \mathbf{u}_n \, \mathrm{d}Q + \int_{\Omega_n} \mathbf{w}^+ \cdot \rho(\mathbf{u}_n^+ - \mathbf{u}_n^-)\, \mathrm{d}\Omega = 0
\end{aligned} \tag{4}$$

where n_{el} is the total number of elements; and δ_1 and δ_2 are two stabilization parameters [3, 4, 6]. The process is applied sequentially to all space-time slabs $Q_1, Q_2, \ldots, Q_N$, starting with $\mathbf{u}_1^- = \mathbf{u}_0$.

2.3. A fully Lagrangian finite element formulation

In the present work, a fully Lagrangian formulation is adopted to describe the motion of fluid flows, together with the use of low-order elements. For a fully Lagrangian description, the mesh will move with the fluid particles. A low-order element considered involves the shape functions which are piecewise linear in space but constant in time.

Under these assumptions, the formulation (4) can be much simplified and the following system of F.E. equations at each space-time slab Q_n can be derived:

$$\begin{bmatrix} \mathbf{M} + \mathbf{K} & \mathbf{C}^{\mathrm{T}} \\ \mathbf{C} & -\mathbf{M}_p \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_p \end{bmatrix} \tag{5}$$

where

$$\mathbf{K}_{ij} = \int_{Q_n} 2\mu\boldsymbol{\varepsilon}(\mathbf{N}_i) : \boldsymbol{\varepsilon}(\mathbf{N}_j)\,\mathrm{d}Q + \int_{Q_n} \rho\delta_2(\nabla\cdot\mathbf{N}_i)\cdot(\nabla\cdot\mathbf{N}_j)\,\mathrm{d}Q$$

$$\mathbf{M}_{ij} = \int_{\Omega_n} \rho\mathbf{N}_i\cdot\mathbf{N}_j\,\mathrm{d}\Omega \qquad \mathbf{C}_{ij} = -\int_{Q_n} N_i\nabla\cdot\mathbf{N}_j\,\mathrm{d}\Omega \qquad \mathbf{M}_{p_{ij}} = \int_{Q_n} \delta_1\nabla N_i\cdot\nabla N_j\,\mathrm{d}Q$$

$$\mathbf{F}_{ui} = \int_{Q_n} \mathbf{f}N_i\,\mathrm{d}\Omega + \int_{Q_n} \rho\mathbf{u}_{n-1}N_i\,\mathrm{d}Q \qquad \mathbf{F}_{pi} = -\int_{Q_n} \delta_1\mathbf{f}\cdot\nabla N_i\,\mathrm{d}Q$$

in which N_i denotes the shape function of node i for both velocity and pressure, and $\mathbf{N}_i = [N_i, \ldots, N_i]^{\mathrm{T}}_{n_{\dim}\times 1}$. Due to the dependency of Q_n on the velocity to be determined, the above equation is essentially a non-linear algebraic system of equations that should be solved iteratively, typically by the Newton–Raphson scheme.

In order to obtain the quadratic rate of asymptotic convergent of the Newton–Raphson iterations, an exact linearization of equations (5) has been achieved in Reference [7]. With a reasonable small time step, our experience shows that generally 2–3 Newton–Raphson iterations are required at each time step for a wide variety of problems.

3. LINEAR SOLUTION APPROACH

By defining a fixed known space-time domain $\bar{Q}_n \in \mathbb{R}^{n_{\dim}} \times I_n$ as the reference configuration, the integration over the domain Q_n in (5) can be rewritten as

$$\int_{Q_n} (\cdot)\,\mathrm{d}Q = \int_{\bar{Q}_n} (\cdot) J_n(\boldsymbol{\xi},\tau)\,\mathrm{d}\bar{Q} = \int_0^{\Delta t_n}\int_{\Omega_n} (\cdot) J_n(\boldsymbol{\xi},\tau)\,\mathrm{d}\Omega\,\mathrm{d}\tau\boldsymbol{\xi} \tag{6}$$

where J_n is the Jacobian which measures the spatial volume transformation ratio from the reference domain to the corresponding physical domain and is a non-linear function of nodal velocities $\mathbf{u}_n$; $\boldsymbol{\xi}$ are the spatial co-ordinates defined in $\bar{Q}_n$, and τ is the local time co-ordinate in the interval $[t_n, t_{n+1}]$. With this integral domain transformation, the source of non-linearity of the equation is transferred from Q_n to J_n.

By exploiting the fact that in each element domain, Ω_n^e, the corresponding Jacobian J_n^e is only a function of local time co-ordinate τ, expression (6) can be further reduced to

$$\int_{Q_n} (\cdot)\,\mathrm{d}Q = \sum_{e=1}^{n_{\mathrm{el}}} \int_0^{\Delta t_n} J_n^e(\tau) \int_{\Omega_n^e} (\cdot)\,\mathrm{d}\Omega\,\mathrm{d}\tau \tag{7}$$

which indicates that J_n^e is integrated over the time domain and is separated from the spatial part.

In an ideal incompressible situation where incompressible condition is fully enforced, J_n, or J_n^e must be equal to 1. In this case, if the integral function $(\cdot)$ is only dependent on the spatial domain, expression (7) becomes

$$\int_{Q_n} (\cdot)\,\mathrm{d}Q = \Delta t_n \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega_n^e} (\cdot)\,\mathrm{d}\Omega = \Delta t_n \int_{\Omega_n} (\cdot)\,\mathrm{d}\Omega \tag{8}$$

which indicates that the non-linearity has completely disappeared. However, as the incompressible constraint is relaxed in the present situation, J_n must be present in the formulation. Nevertheless, it is reasonable to expect that J_n may be sufficiently close to 1 if an acceptable level of solution accuracy can be achieved at each time instant.

This observation motivates a possibility to derive an approximate solution scheme by explicitly setting J to be 1 in the formulation. Consequently, by assuming that the body force $\mathbf{f}$ is not time-dependent, together with the fact that the shape functions $\mathbf{N}$ are only the function of (reference) spatial co-ordinations ξ, Equation (5) can be rewritten as

$$\begin{bmatrix} \mathbf{M}/\Delta t_n + \bar{\mathbf{K}} & \bar{\mathbf{C}}^{\mathrm{T}} \\ \bar{\mathbf{C}} & \bar{\mathbf{M}}_p \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{F}}_u \\ \bar{\mathbf{F}}_p \end{bmatrix} \tag{9}$$

where $\mathbf{M}$ is given in (5) and

$$\bar{\mathbf{K}}_{ij} = \int_{\Omega_n} 2\mu\boldsymbol{\varepsilon}(\mathbf{N}_i) : \boldsymbol{\varepsilon}(\mathbf{N}_j)\,\mathrm{d}\Omega + \int_{\Omega_n} \delta_2(\nabla\cdot\mathbf{N}_i)\cdot(\nabla\cdot\mathbf{N}_j)\,\mathrm{d}\Omega$$

$$\bar{\mathbf{C}}_{ij} = -\int_{\Omega_n} N_i\nabla\cdot\mathbf{N}_j\,\mathrm{d}\Omega \qquad \bar{\mathbf{M}}_{p_{ij}} = \int_{\Omega_n} \delta_1\nabla N_i\cdot\nabla N_j\,\mathrm{d}\Omega$$

$$\bar{\mathbf{F}}_{ui} = \int_{\Omega_n} \mathbf{f}N_i\,\mathrm{d}\Omega + \int_{\Omega_n} \rho\mathbf{u}_{n-1}N_i\,\mathrm{d}\Omega \qquad \bar{\mathbf{F}}_{pi} = -\int_{\Omega_n} \delta_1\mathbf{f}\cdot\nabla N_i\,\mathrm{d}\Omega$$

Clearly this system of equations is linear. Therefore at each slab only a linear Stokes-like problem needs to be solved. Consequently the computational costs can be reduced by 2–3 times in comparison with the Newton–Raphson non-linear scheme. In addition, the stiffness matrix will become symmetric and this feature can further reduce the costs. Such a scheme may be identified as a forward Euler procedure, which may impose a condition on the time-step size. However, as the time-step size is usually required to be sufficiently small in order to achieve a reasonable solution accuracy, no stability problem has been observed in our numerical experience.

4. MESH ADAPTIVITY

Numerical simulation of fluid flows involving moving domains is often characterized by the substantial deformation. Without an appropriate scheme to handle the deformation of the mesh, the solution procedure will very rapidly run into difficulty due to severe element distortion or even element tangling. Therefore, the introduction of spatially adaptive remeshing processes is crucial for the successful solution of such engineering problems. Another objective of employing an adaptive re-meshing is to optimize the element distribution according to the feature of the intermediate solution in order to achieve a targeted solution accuracy with a minimal number of elements.

As a fully Lagrangian description for fluid flow is adopted, the relatively matured mesh adaptivity technique developed for modelling large deformation problems in solid mechanics can be employed without any fundamental modifications. In the present work a mesh-size indicator based on the total strain rate is employed. An unstructured *advancing front* technique is used for the mesh generation and subsequent mesh adaptation, which allows a relatively simple control of the mesh density and geometric recovery of the deformed surfaces. Details can be found, for instance, in References [8, 9].

5. NUMERICAL ILLUSTRATION

The applicability of the above linear solution strategy, together with the mesh adaptivity technique is illustrated by a wave propagation and breaking problem [10]. The initial geometry of the problem is shown in Figure 1, with $L = 100\,\text{m}$, $H = 10\,\text{m}$, $d = 5\,\text{m}$, $d_1 = 2\,\text{m}$, $d_2 = 0.857\,\text{m}$. The slope of the shoaling bottom is taken to be $\frac{1}{14}$. The initial surface elevation, velocities and pressure of the wave are determined by Laitone's solution for a solitary wave of finite amplitude propagating without change of shape [11]. The wave crest is located at a distance of $L/2$ from the left end. A stick boundary condition is assumed for the shoaling bottom. Water is assumed viscous with a viscosity of $\mu = 1.01 \times 10^{-3}\,\text{Ns/m}^2$ and a density of $\rho = 1000\,\text{kg/m}^3$. The acceleration of gravity is set to be $9.8\,\text{m/s}^2$. Two stabilization parameters, δ_1 and δ_2, in the formulation (9) are chosen to be 10^{-11} and 1, respectively. A very small value of δ_1 ensures that the volume change in the simulation remains at a very low level.

Figures 2(a)–2(d) illustrate the deformed and adapted finite element configurations of the wave at 4 different time instants, while Figures 3(a)–3(d) show the corresponding velocity distributions in the horizontal direction. Different time steps are tested and the main difference appears in the form of a slight difference in the time instant at which the wave is breaking. A smaller step size leads to a slightly shorter time. This is mainly due to the numerical damping associated with the time integration scheme employed. Refining the mesh has a similar effect. For $\Delta t = 10^{-3}$ with the meshes shown in the figures, the breaking time is 9.65 s, which closely matches the result of 9.9 s obtained in Reference [10]. The simulation is also conducted by employing the non-linear Newton–Raphson procedure and the difference in results from the described linear procedure is negligible (less than 0.1%). The time adaptivity as described in Reference [7] is not adopted in this case. It is noted that the current solution procedure requires further developments in order to allow modelling of surface merging after the wave breaking.

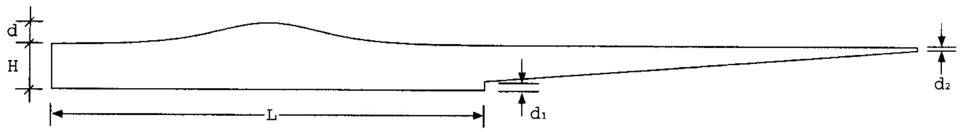

Figure 1. Initial geometry of the wave propagation and breaking problem.

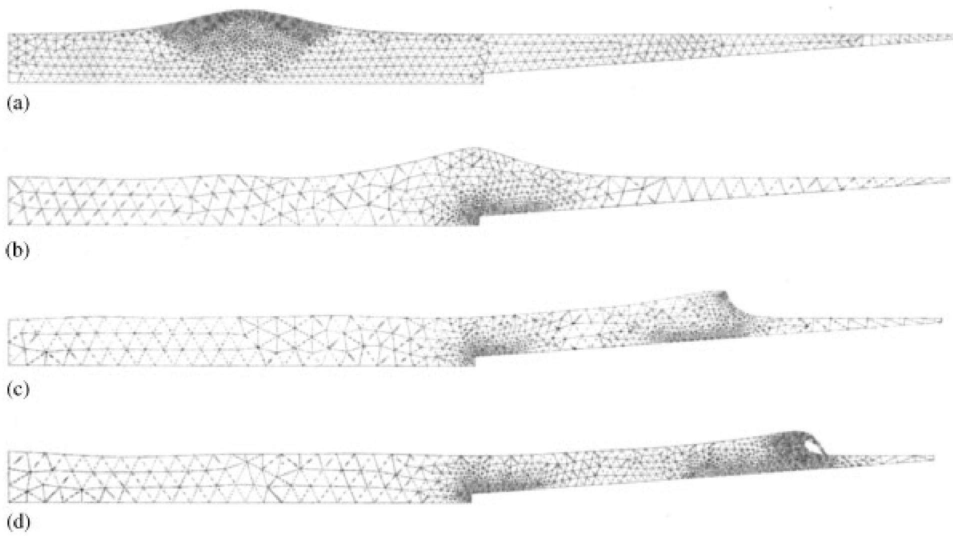

Figure 2. Adapted deformed configurations at 4 different time instants: (a) $t = 0.0$ s; (b) $t = 4.0$ s; (c) $t = 8.0$ s; (d) $t = 9.65$ s.

Figure 3. Horizontal velocity distributions at 4 different time instants: (a) $t = 0.0$ s; (b) $t = 4.0$ s; (c) $t = 8.0$ s; (d) $t = 9.65$ s.

6. CONCLUSIONS

A linear solution strategy for the finite element simulation of incompressible fluid flows with moving domains is outlined in the context of a fully Lagrangian space-time GLS formulation using low-order elements. Its applicability is demonstrated for a wave propagation and breaking problem. It has been demonstrated that for this type of problems adaptive re-meshing techniques are essential to achieve a successful simulation.

ACKNOWLEDGEMENTS

This work is funded by the EPSRC of UK under grant No. GR/L66724. This support is gratefully acknowledged.

REFERENCES

1. Tezduyar TE, Behr M, Liou J. A new strategy for finite element computations involving moving boundaries and interfaces—the deforming-spatial-domain/space-time procedure: I. The concept and the preliminary numerical tests. *Computer Methods in Applied Mechanics and Engineering* 1992; **94**:339–351.
2. Tezduyar TE, Behr M, Mittal S, Liou J. A new strategy for finite element computations involving moving boundaries and interfaces — the deforming-spatial-domain/space-time procedure: II. Computation of free-surface flows, two-liquid flows and flows with drifting cylinders. *Computer Methods in Applied Mechanics and Engineering* 1992; **94**:353–371.
3. Behr M, Tezduyar TE. Finite element solution strategies for large-scale flow simulations. *Computer Methods in Applied Mechanics and Engineering* 1994; **112**:3–24.
4. Hansbo P. Lagrangian incompressible flow computations in three dimensions by use of space-time finite elements. *International Journal for Numerical Methods in Fluids* 1995; **20**:989–1001.
5. Hughes TJR, Franca LP, Hulbert GM. A new finite element formulation for computational fluid dynamics: VIII. The Galerkin/least-squares method for advective-diffusion equations. *Computer Methods in Applied Mechanics and Engineering* 1989; **73**:173–189.
6. Masud A, Hughes TJR. A space-time Galerkin/least-squares finite element formulation of the Navier–Stokes equations for moving domain problems. *Computer Methods in Applied Mechanics and Engineering* 1997; **146**:91–126.
7. Feng YT, Perić D. A time adaptive space-time finite element method for incompressible Lagrangian flows with free surfaces: computational issues. *Computer Methods in Applied Mechanics and Engineering* 2000; **190**(5–7):499–518.
8. Feng YT, Perić D. Coarse mesh evolution strategies in the Galerkin multigrid method with adaptive remeshing for geometrically nonlinear problems. *International Journal for Numerical Methods in Engineering* 2000; **49**(4):547–571.
9. Perić D, Vaz Jr M, Owen DRJ. On adaptive strategies for large deformations of elasto–plastic solids at finite strains: computational issues and industrial applications. *Computer Methods in Applied Mechanics and Engineering* 1999; **176**:279–312.
10. Radovitzky RA, Ortiz M. Lagrangian finite element analysis of Newtonian fluid flows. *International Journal for Numerical Methods in Engineering* 1998; **43**(4):607–619.
11. Ramaswamy B, Kawahara M. Lagrangian finite element analysis applied to viscous free surface fluid flow. *International Journal for Numerical Methods in Fluids* 1987; **7**:953–984.

